Consolidated Week 6 Notes MAST30026 Sep 14, 2014

Lecture 1:

* **Def^{n}:** Let (X, d) be a metric space. $A \subseteq X$ is a *dense* subset if

(a) closure of $A = X$

(b) $X = A \cup \{\text{limit points of } A\}$

(c) if $U \subseteq X$ is open & non-empty then $U \cap A \neq \emptyset$.

** **Theorem:** (Baire's Theorem) Let (X, d) be a complete metric space.

Let U_i, $i = 1, 2, 3, \ldots$ be open dense sets in X.

$\Longrightarrow \bigcap_{i=1}^{\infty} U_i$ is dense in X.

Lecture 2:

** **Theorem:** U_i is open dense in X complete $\Longrightarrow \bigcap_{i=1}^{\infty} U_i$ is dense.

** **Theorem:** (Baire's Theorem Second Version)

Recall: A set $A \subseteq X$ is nowhere dense if interior $\overline{A} = \emptyset$.

So if U is any non-empty open set then $U \not\subseteq \overline{A}$.

Recall: interior $\overline{A}$ = "biggest/maximum open set inside $\overline{A}$"

$\Longleftrightarrow U \cap (X \setminus \overline{A}) \neq \emptyset$

$\Longleftrightarrow X \setminus \overline{A}$ is dense.

Let (X, d) be a complete metric space. Let F_i be a sequence of nowhere dense sets, $i = 1, 2, 3, \ldots$

Then $\bigcup_{i=1}^{\infty} F_i$ has empty interior.

- Example: F_i = circle in $\mathbb{R}^2$

Easy to show F_i is closed ($\mathbb{R}^2 \setminus F_i$ is open).

$\text{int}(\overline{F_i}) = \text{int}(F_i) = \emptyset$

$F_i = \{x \in \mathbb{R}^2 \mid d(x, x) = r\}$

No open ball lies in F_i

Conclusion: $F_1 \cup F_2 \cup \cdots$ has no interior.

In particular, $F_1 \cup F_2 \cup \cdots \neq \mathbb{R}^2$

Lecture 3:

* **Def$^{\underline{n}}$:** $(V, \|\cdot\|)$ is a **Banach space** if V is a vector space (concentrate on scalars $\mathbb{R}$ or $\mathbb{C}$), $\|\cdot\|$ is a norm and the induced metric d is complete.

Recall: $\|\cdot\|$ is a norm, so (a) $\|v\| \geq 0$, $\|v\| = 0$ if and only if $v = 0$

(b) $\|\lambda v\| = |\lambda| \cdot \|v\|$

(c) $\|v + v'\| \leq \|v\| + \|v'\|$

Define $d(v, v') = \|v - v'\|$.

If v_n is a Cauchy sequence

$$\|v_n - v_m\| = d(v_n, v_m) \longrightarrow 0 \text{ as } n, m \longrightarrow \infty$$

$$\Longrightarrow v_n \longrightarrow v^* \text{ in } V$$

– **Examples:** $(V, \|\cdot\|)$, with a good choice of norm, is alway Banach if V is finite dimensional.

(1) $\mathbb{R}^n$ or $\mathbb{C}^n$ with various norms are Banach

p-norm: $\|(a_1, a_2, \dots, a_n)\|_p = \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p}$, $p \geq 1$.

∞-norm: $\|(a_1, a_2, \dots, a_n)\|_\infty = \sup_{i=1,n} |a_i|$.

(2) ∞-dimensional versions, sequence spaces.

$\ell^p = \{(a_1, a_2, \dots) \mid a_i \in \mathbb{R} \text{ or } \mathbb{C}, \sum_{i=1}^{\infty} |a_i|^p < \infty \text{ (converges)}\}$

$\|(a_1, a_2, \dots)\|_p = \left(\sum_{i=1}^{\infty} |a_i|^p\right)^{1/p}$, $p \geq 1$.

$\ell^\infty = \{(a_1, a_2, \dots) \mid a_i \in \mathbb{R} \text{ or } \mathbb{C}, \sup |a_i| < \infty \text{ (bounded sequence)}\}$

$\|(a_1, a_2, \dots)\|_\infty = \sup_i |a_i|$.

So ℓ^p, ℓ^∞ are Banach spaces.

(3) $C_b(X, \mathbb{R} \text{ or } \mathbb{C})$ = space of continuous bounded functions from X to $\mathbb{R}$ or $\mathbb{C}$.

X can be a topological space as well as just a metric space.

and $\|f\| = \sup_{x \in X} |f(x)|$ (Extends $(\mathbb{R}^n, \|\cdot\|_\infty)$ & ℓ^∞).

* **Def$^{\underline{n}}$:** A series $\sum_{i=1}^{\infty} \lambda_i v_i$ is **norm absolutely convergent** if $\sum_{i=1}^{\infty} \|\lambda_i v_i\|$ converges.

** **Theorem:** If $(V, \|\cdot\|)$ is any normed vector space then the metric is complete if and only if every norm absolutely convergent series is convergent.